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PARTS V & VI] SECTION A [VOL. 18

ABSORPTION MEASUREMENTS AT LOW FREQUENCIES IN SOME LIQUIDS

By

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[Communicated by Dr. R. N. Ghosh, F. N. I., F. A. Sc. (America)]

(Received on June 2, 1949)

INTRODUCTION

Absorption measurements in liquids have generally been made at frequencies ranging from 4 to 15 megacycles where the absorption is fairly large. We have studied the absorption in Benzene, carbon-tetra-chloride, transformer oil and acetic acid at 1·46 and 4 megacycles with the help of an Ultrasonic Interferometer.

It is well known that when elastic waves travel in a liquid medium, an attenuation is produced due to viscosity and heat conduction which has been worked out by Stokes (1845) and Kirchoff (1868). Value of attenuation coefficient ranging from 1500 to 3 times the calculated values have been observed [Willard (1941) Pellam (1946)] in some unassociated liquids in the range from 4 to 15 megacycles but attenuation coefficient is found to be strictly proportional to the square of the frequency in agreement with the formula of Stokes and Kirchoff. In highly viscous liquids Hunter (1941) finds that the observed value of attenuation coefficient is the same as the calculated one.

It may be stated that Stokes and Kirchoff's formula becomes inadequate in the neighbourhood of absorption frequency although

it remains true for frequencies much larger than the absorption frequency.

We have verified these results for frequencies 1·46 and 4 Mcs. where the absorption is low and difficult to be measured. Transformer oil which has been studied for the first time is fairly viscous liquid ($\mu = \cdot 051$) and we have been able to verify the results given by Hunter. For Acetic Acid our results agree with those obtained from the theoretical curve of α/N against N, where N is the frequency of the sound wave, given by Pinkerton (1948).

DESCRIPTION AND WORKING OF THE APPARATUS

A quartz plate which is driven by an alternating e.m.f. of constant amplitude acts as a source of sound and is placed in the lower portion of a cylindrical vessel in such a way that its radiating face is parallel and opposite to a movable reflector which can be moved by means of a micrometer screw. As the reflector is moved towards or away from the source, the current in the standard circuit to which the crystal plate is connected in parallel shows periodic changes which are easily measurable with a thermocouple meter. At resonance points the meter shows sharp peaks situated one-half acoustic wavelength apart. As the frequency can be determined with the help of hetrodyne wavemeter, the velocity of sound becomes known. The absorption coefficient is determined from the decrement of the reaction.

The chamber wall is a section of brass tubing 1/16″ thick, 4″ in diameter and 10″ high. The crystal support and the micrometer screw with its attached reflector were made a part of the upper plate in order that careful alignment of the crystal and reflector could be made before it was placed in the chamber.

The crystal mounting as shown in (Fig. 1) was designed to hold the crystal in place with a minimum of mechanical friction to its oscillation and to provide an insulated support for leading the radio-frequency driving voltage to the lower surface of the crystal.

The lower electrode which consists of a plane rectangular brass strip rests in a ebonite cavity and soldered to the lead in the middle. The upper electrode is earthed. The whole mounting is provided with three levelling screws to maintain parallelism between upper surface of the crystal and the reflector.

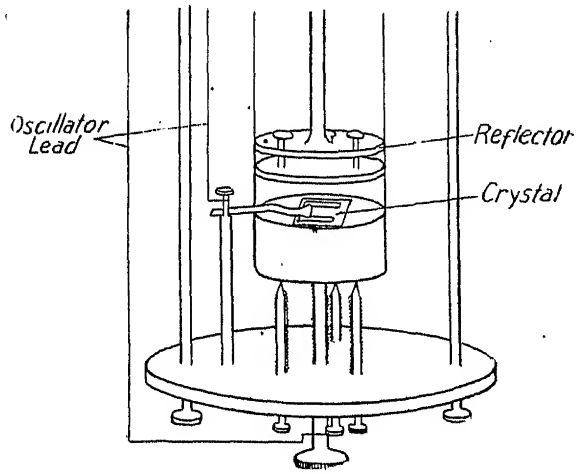

Figure 1

The driving frequency stabilised oscillator, buffer amplifier and the resonance circuits as shown in (Fig. 2) are the standard ones. The oscillator is a frequency stabilised crystal-controlled oscillator. The amplifier is a radio frequency tuned plate amplifier the input of which is coupled to the tank circuit through a condenser of ·001 μfd. The advantage of having a buffer amplifier in between the oscillator and the interferometer is the reduction of the effects of load variations on the frequency of the oscillator. The power at which absorption is to be studied can be studied at will by varying the coupling between the resonance circuit and the output of the amplifier.

A series of observations recording maxima and minima of current for various positions of the reflector as it is moved away from the source are taken. Let $\sigma = (i/I)$ where 'i' is the value of current in the output resonance circuit and I is the maximum value of 'i' at resonance with interferometer disconnected. A curve is plotted between σ and the distance of the reflector from the source. The curve shows a series of

values of σ_{min} situated one-half acoustic wavelength apart. The attenuation coefficient a may be calculated from the relation given by Fox (1937) between the values of r and σ_{min}

$$ra = \frac{c}{s} \cdot \frac{\sigma_{min}}{1-\sigma_{min}} + \text{const}, \tag{1}$$

where a is the attenuation coefficient and c and s are interferrometer constants which can be determined. Plotting $\sigma_{min}/(1-\sigma_{min})$ against r we get a st. line whose slope gives the value of a.

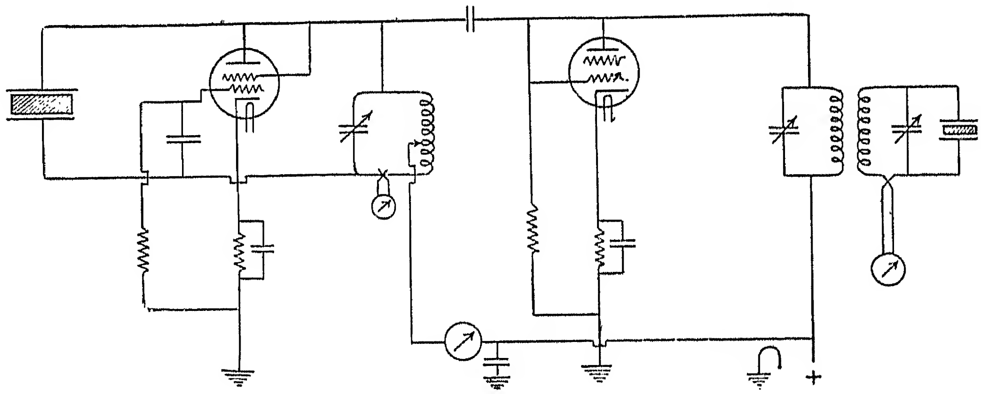

Figure 2

The value of σ_{min} depends upon the parallelism between crystal and the reflector. It may be argued that since we are measuring very small distances of the order ·005 cm, a lack of parallelism between the crystal and the reflector due to motion of the latter will introduce errors in the value of σ_{min}, To check the source of error due to this cause we have obtained the values of current for any position of the reflector for serveral independent settings. These are given in Table 1 and it may be seen that the maximum variation in the current does not exceed ·7%. We have therefore taken the mean for several such observations to obtain the value of 'i'. σ_{min} has been (fig. 3a) obtained by plotting σ (i/I) and r. The minimas are quite sharp. The points on the curve between $\frac{\sigma_{min}}{1-\sigma_{min}}$ and r fall almost on a st. line as shown in Fig. (3b). This also shows that the error due to lack of parallelism

as the reflector is moved is negligible small. Table 1 gives the observations for Benzene.

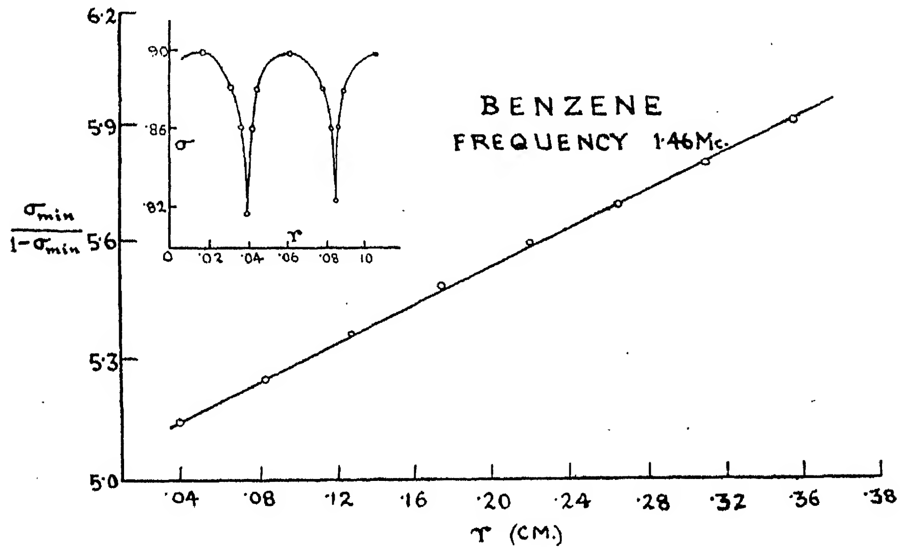

Fig. 3(*a*)

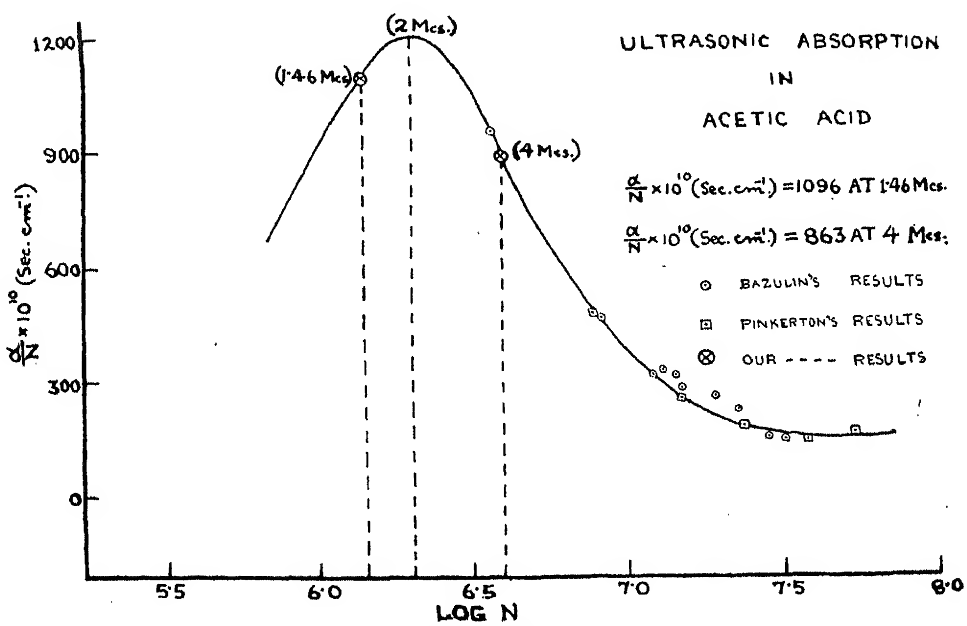

Fig. 3(*b*)

TABLE 1

Least count of micrometer screw ·0002 *cm*.

'*i*' current at minimas

'*r*' distance of the reflector from the crystal.

r (Cm)	·0400	·0852	·1304	·1757	·2209	·2661	·3113	·3566
'i' 'Ma' a	361	363	366	369	372	374	·377	380
,, 'b'	360	364	366	368	372	375	·377	381
,, c	360	363	367	370	373	375	·377	380
,, d	361	364	366	369	373	375	376	381
,, e	360	363	367	369	372	375	377	380
,, f	360	364	366	369	372	375	376	380
Mean	360·3	363·5	366·3	369	372·3	374·8	376·6	380·3
$\lambda/2$	226 Div	228 Div	226 Div	226 Div	226 Div	226 Div	226 Div	226 Div

RESULTS AND DISCUSSIONS

The following table shows the values of absorption as well as the velocity of sound waves eperimentally obtained in Benzene, carbon-tetra-chloride, transformer oil and Acetic Acid at 1·46 and 4 Mcs. with the help of formula (1). The wavelength has been measured as the distances between two successive minimas.

TABLE 2

Liquid	Freq.	Temp.	λ Measured	α (Cm⁻¹)	$\alpha/N^2 \times 10^{17}$ Sec² cm⁻¹	Velocity metre/Sec	Values of α/N^2 obtained by others	Reference
Benzene	1·46Mc.	23°C	·0904 cm	·02	940	1320	923 at 207 Mc.	Gregg (1941)
,,	4 Mcs.	23·2°C	·0905 cm	·15	937	1321	863 at 477 Mcs.	Baumgardt (1937)
Carbon-tetra-chloride	1·46Mc.	23·6°C	·0635 cm	·011	507	928	—	
,,	4 Mcs.	24°C	·0637 cm	·083	520	980	553 at 4·4 Mc	Willard (1941)
Trans-former oil	1·46Mc.	23·4°C	·0879 cm	·0021	104	1284		
,,	4 Mcs.	23·2°C	·0880 cm	·0169	106	1286		
Acetic Acid	1·46Mc.	23°C	·0843 cm	·16	7619	1231	extra-polated	Pinketron (1948)
,,	4 Mcs.	23·3°C	·0858	·34	2150	1253	2240 at 24 Mc.	Pinkerton (1948)

Benzene has been worked by several workers namely Gregg (1941), Baumgardt (1937). Claeys Errera & Sack (1936), Rieckmann (1939), Bazulin (1937) Lindberg (1940), and Pellam and Galt (1946). The values of $(\alpha/N) \times 10^{17}$ for Benzene obtained by different workers vary from 800-1157, although a large number of them have found a value in the neighbourhood of 920. As no frquency effect on absorption can be expected at such low frequencies, this difference appears to be due to impurity of Benzene and experimental errors. The values of α/N^2 quoetd in the last column have been chosen because the author worked close to the frequencies used by us.

For carbon-tetra-chloride, Biquard (1935) and Pellam (1946) give a value close to 530, while Willard gives a value of 570 and

Parthasarthy—421, 492. 632. Our value is close to Biquard and Pellam.

Transformer oil has not been worked out previously.

Acetic Acid has been studied by Bazulin (1935-37) and Pinkerton (1948) at several frequencies varying from 4 Mcs. to 15 Mcs. and they have plotted a curve between α/N and log N(Fig. 4) where N is the frequency of the sound wave. The curve is almost a st. line in the region 30 Mcs. to 100 Mcs, the value of $(\alpha/N^2)\times 10^{17}$ being 160. For frequencies below 30 Mcs. t e value shoots up, being 400 at 15 Mcs, 600 at 10 Mcs., 1000 at 7 Mcs., and 2240 at 4 Mcs. They have further extrapolated the curve below 4 Mcs. by using the formula of Kneser (1938) and thus they show that there is an absorption frequency close to 2 Mcs. Our value of $(\alpha/N^2)\times 10^{17}$ at 4 Mcs. agrees with their experimental values as obtained from their experimental curve. The value at 1·46 Mc. obtained by us also agree with the value obtained from their extrapolated curve.

Using the Stokes' and Kirchoff's formula we have calculated the value of $(\alpha/N^2)\times 10^{17}$ for transformer oil. It comes out to be 96 in good agreement with our observed value. This shows that the viscosity effect is most predominant in sound absorption in transformer oil and that the Stoke's formula holds good for viscous substances ($\mu = \cdot 051$)

ACKNOWLEDGEMENT

I offer my sincerest thanks to Dr. R. N. Ghosh, F. N. I., F. A. Sc. (America) for his valuable guidance throughout the progress of the research as well as to Council of Scientific and Industrial Research for a research grant.

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THEORY OF THE CLARIONET

By

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Some eleven years ago the author published a paper (1) (1938) under the above heading; since then considerable amount of experimental analysis of the tone quality in different registers has appeared. These results combined with Miller's Classical work (2) (1926) have enabled the author to check the theory, and the present paper gives an account of the theoretical work in the light of the experimental work. Considerable light is thrown on the performance of the reed with regard to the tone quality and the presence of even harmonics. We assume as before a uniform of the air chamber, length l, radius a, and neglect radial vibrations in the pipe. We represent the solution of the wave equation for plane waves for excess pressure p by Heaviside's method at any point x measured from the mouth *as*

$$p_x = \eta_1 \sinh \frac{\sigma}{c} x + \eta_2 \cosh \frac{\sigma}{c} x \quad \ldots \quad (1)$$

where $\sigma = \frac{d}{dt}$; c velocity of sound. The particle velocity u is given by

$$\sigma \rho u = -\frac{dp}{dx}$$

i.e.

$$u = -\frac{1}{c\rho}\left\{\eta_1 \cosh \frac{\sigma}{c} x + \eta_2 \sinh \frac{\sigma}{c} x\right\}.$$

Hence at $x=0$, $p=\eta_2$, and $u=-\eta_1/c\rho \quad \ldots \quad (2)$

and,

$$\frac{p_1}{u_1} = -c\rho \left\{ \frac{\eta_1 \sinh \frac{\sigma}{c} l + \eta_2 \cosh \frac{\sigma}{c} l}{\eta_1 \cosh \frac{\sigma}{c} l + \eta_2 \sinh \frac{\sigma}{c} l} \right\}$$

where p_1 and u_1, represent the pressure and particle velocity at $x=1$ the open end. The impedance due to the waves that spread out in the open atmosphere is represented for small values of ka where $k = 2\pi/\lambda$—the where λ is the wave length,

$$\mathcal{Z}=c\rho\{R+jX\} \qquad \ldots \quad (4)$$
$$R=k^2a^2/2;\ X=8ka/3\pi$$

Writing R' for $(R+jX)$ for the sake of brevity we find

$$\frac{\eta_1}{\dot{\eta}_1}=-\frac{R'\sinh\frac{\sigma}{c}l+\cosh\frac{\sigma}{c}l}{R'\cosh\frac{\sigma}{c}l+\sinh\frac{\sigma}{c}l} \qquad \ldots \quad (5)$$

on equating (3) and (4). The flow velocity u through the chink in the reed will be given by

$$u=K(P-p)(y+y_0) \qquad \ldots \quad (6)$$

where P is the blowing pressure, p the pressure in the month; y_0 represents a permanent displacement of the reed; this represents a permanent opening, and y the varibale displacement measured positively outwards; K—conductivity of the opening. K will have the dimensions of (LT/M), and is supposed to be small. A complete solution of (5) in combination with (6) is very difficult; at the first instance we neglect y in (6) in comparison with y_0 we assume y to be small with respect to y_0, then on substituting the values of p and u from (2) we get

$$-\frac{\eta_1}{K'c\rho}=P-\eta_2, \quad K'=Ky_0$$

And substituting the value of from (5) we get

$$\eta_2=\frac{P}{\left\{1+\frac{1}{K'c\rho}\cdot\frac{R'\sinh\frac{\sigma}{c}l+\cosh\frac{\sigma}{c}l}{R'\cosh\frac{\sigma}{c}l+\sinh\frac{\sigma}{c}l}\right\}}$$

This represents Heaviside's solution, and it will be interprete! by the expansion theorem, according to which η_2 will be given by

$$\eta_2=\frac{Y(0)}{\mathcal{Z}(0)}+\sum\frac{e^{\sigma kt}}{\sigma K\,\mathcal{Z}'(\sigma k)}$$

$$Y(0)=1;\ \mathcal{Z}(\sigma)=\left\{1+\frac{1}{K'c\rho}\cdot\frac{R'\sinh\frac{\sigma}{c}l+\cosh\frac{\sigma}{c}l}{R'\cosh\frac{\sigma}{c}l+\sinh\frac{\sigma}{c}l}\right\}$$

and $Z'(\sigma_k)=\frac{dZ(\sigma)}{d\sigma}\ \sigma=\sigma_k$, σ_k being the roots of

$$\left\{1+\frac{1}{K'c\rho}\frac{R'\sinh\frac{\sigma}{c}l+\cosh\frac{\sigma}{c}l}{R'\cosh\frac{\sigma}{c}l+\sinh\frac{\sigma}{c}l}\right\}.$$

Two cases arise (1) if K' is large i. e., the mouth is wide open, then we get the imaginary roots

$$\sin\frac{\omega(l+\alpha)}{c}=0$$

where $\tan\frac{\omega\alpha}{c}=X$; this gives $\omega l'/c=m\pi$, where m is any integer, and $l'=(l+\alpha)$.

If K' is small then we find $\cos(\omega l'/c)=0$

i.e. $\omega l'/c=\frac{m\pi}{2}$, m being an odd integer, and $\lambda_m=4l/m$.

We write $$R+jX=\beta^2m^2+j\alpha m$$

$$\beta^2=\pi^2a^2/8l^2;\ \alpha=(4a/3l)$$

where a is the radius of the pipe. This holds good for small values of $ka<1$

$$\frac{Y(0)}{Z(0)}=Q(\beta^2m^2+j\alpha m)(y'+y)$$

$$y'=(y_0-AP/Mn^2),\ Q=Kc\rho.$$

Hence $$p=PQ\left[S_0+(\beta^2m^2+\alpha jm)y'+y(\beta^2m^2+j\alpha m)+\frac{cy'}{L}\sum\frac{e^{jm\omega_1 t}}{jm\omega_1}\right.$$

$$\omega_1=\pi c/2l' \tag{11}$$

S_0 represents the sum of a finite number of terms in m^2. This sum will have small value. In order to determine y we observe that the equation of motion of the reed

$$(\sigma^2+n^2)y=\frac{-(P-p)A}{M}$$

where A effective surface of the reed, and M its effective mass $n/2\pi$ redresents its natural frequency.

Hence $$y=-\frac{AP}{Mn^2}+\frac{Q_1cy'}{l}\sum\frac{e^{jm\omega_1 t}}{jm\omega_1(n^2-m^2\omega_1^2)} \tag{12}$$

where $Q_1=APQ/M$. substituting this value of y in (11) we get

$$p = PQ\left[S_1 + \frac{Q_1\beta^2 cy'}{l}\sum\frac{m^2 e^{jm\omega_1 t}}{jm\omega_1(n^2-m^2\omega_1^2)} + \frac{Q_1\alpha cy'}{l\omega_1}\sum\frac{e^{jm\omega_1 t}}{(n^2-m^2\omega_1^2} + \frac{cy'}{l}\sum\frac{e^{jm\omega_1 t}}{jm\omega_1} - \text{weak even harmonics} - \frac{Q_1 c^2 y'}{9l^2\omega_1^2}\frac{e^{10j\omega_1 t}}{(n^2-9^2\omega_1^2)}\right]$$

where it is assumed $n \sim 9\omega_1$, S_1 represents the sum of all the possible terms that are not periodic. The terms containing β^2 may be neglected. Similarly in the cosine series the terms containing $(n^2-m^2\omega_1^2)$, in the denominator where $n \sim m\omega_1$, will be important and have to be retained and the rest may be neglected. Thus finally we shall have

$$p = PQ\left[S_1 + \frac{\alpha cy'}{l\omega_1}\sum\frac{\cos m\omega_1 t}{(n^2-m^2\omega_1^2)} + \frac{cy'}{l}\sum\frac{\sin m\omega_1 t}{m\omega_1} - \text{weak even harmonics} - \frac{Q_1 c^2 y'}{9l^2\omega_1^2}\frac{e^{10j\omega_1 t}}{(n^2-9^2\omega_1^2)}\right]$$

It will be noticed that the coupling of the chink through which air is blown at a constant pressure with the air column is sufficient to maintain the vibration of the air column. In the case of a vibrating reed having a very high natural frequeucy of vibration it does help in the manner expressed by the term $\sum\frac{\cos m\omega_1 t}{(n^2-m^2\omega_1^2)}$. This means that at the initial stage the compressional wave starts with the chink partially widened and when it reaches back after reflection from the open end as rarefaction, the reed has performed four and a half vibration and it has a negative displacement i.e., towards the flat table closing the chink partially, This helps the reflection of the rarefaction from the partially closed end; at this instant the coupling of the mouth piece with the vibrating air column has suddenly altered the pressure to a negative value. When the rarefaction reaches back as compression after reflection from the open end, the reed has performed another four and a half vibration, and at the instant the reed has positive displacement i.e., the chink is widened. The widening of the chink allows more air to enter the mouth piece and the compressional pressure is increased. It will be noticed that the reed performs forced vibration at a frequency $n \sim m\omega_1$, and in order to maintain the

vibration, this condition has to be realised. The reed has only one natural frequency on account of its chiseled shape. In practice the musicians adjust the frequency with care so that the frequency of reed is in agreement with some odd component of the clarionet.

The flow of air is obtained easily from the results that

$$u=\frac{R}{c\rho}\,p=\frac{\beta^2 m^2}{c\rho}p$$

$$u\sim PQ\left[S_2+\frac{\beta^2 cy'}{l}\sum\frac{m^2\cos m\,\omega_1 t}{(n^2-m^2\omega_1^2)}+\frac{cy'}{l}\sum\frac{m^2\sin m\omega_1 t}{m\omega_1}\right].$$

Since we sum only upto $m=1$, the summation will not give us infinitely large value. It will be noticed that the second term will vanish in the absence of vibration of the reed, and then the coupling term

$$\sum\frac{m^2\sin m\,\omega_1 t}{m\,\omega_1}$$

will remain and will be sufficient to maintain the vibration of the air column. The presence of m^2 in the numerator lends to the belief that the flow velocity series will become divergent i. e. will give a very large value by taking m large. Since we sum up to $m=1$ i.e. 6 terms, the presence of the factor β^2 does not allow the summation to obtain infinitely large value for the sum. These results hold good as long as $ka<1$. when $ka\geqslant 1$ then z in (4) will be more complex, and all our formulae will fail to represent the true facts. In actual prectice harmonics up to $m=11$, have been analysed and for pipes of small bore $ka<1$, and then the summatian of the important terms will give a small finite value.

The effect of vibration of the reed is mainly upon the quality of the note. It will be noticed that in the absence of vibration of the need, the pressure changes will practically be given by the sum $\Sigma\sin m\,\omega_1 t/m\,\omega_1$. This summation will give nearly a constant positive pressure with small fluctuations when six or seven terms are considered in the summation viz from m 1 to m 11, over a half cycle, and then suddenly changes to a negative value with similar fluctuations. At the end of the complete cycle, the pressure again rises suddenly

to a positive value. These changes are practically discontinuous. But the vibration of the reed through the terms

$$\sum \frac{\cos m\, \omega_1 t}{(n^2 - m^2\, \omega_1{}^2)}$$

introduces larger fluctuations of pressure within the half cycle. The near n is to $m\omega_1$, the greater will be the modulations of pressure and hence more of harmonics will elicited near $n \sim m\omega_1$. In general n is large-er say $n \sim 9\omega_1$; under these circumstances all those odd components near $9\omega_1$ will be excited. It is possible to fix the value of n by adjusting he vibrating length of the reed, or by slight alteration of the same by the pressure of the lips upon the reed. Thus it will be noticed that player can control the ton quality at pleasure.

The presence of weak even harmonics or a strong component corresponding to the natural frequency of the reed is easily brought about by the product terms which contain an abundance of weak even harmonics. Another point to be noticed is the dependence of the amplitude of the components on Y'

i. e. $(Y_0 - AP/Mn^2)$

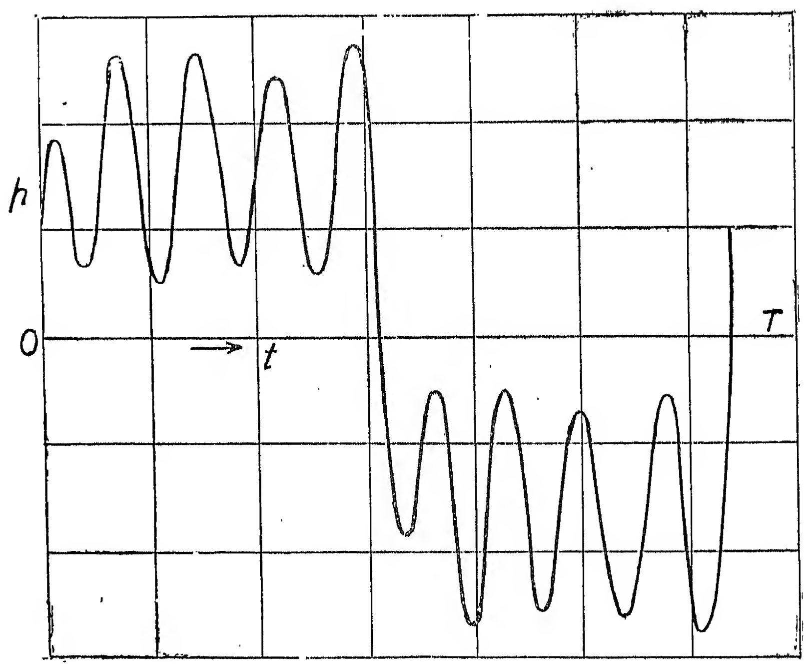

Figure 1

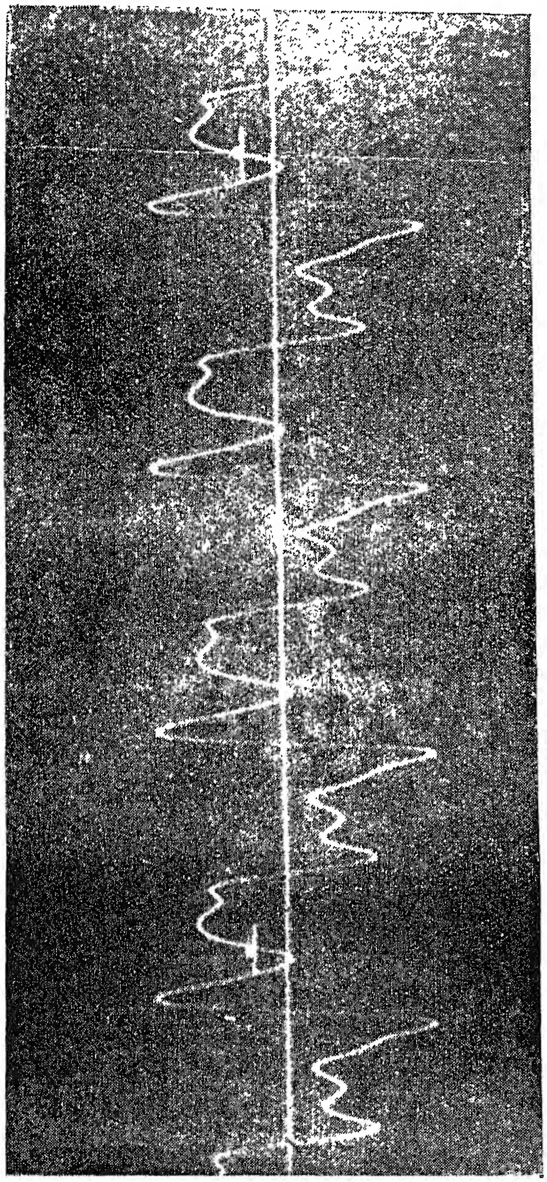

Figure 2

which must be positive to ensure the adjustment of phase favourable for the maintenance of vibration of the air column.

Comparsion of the pressure time curve with that of prof. Millers's curve for the clarionet shows great similarity. Tne latter curve shows a discontinuous change at the end of a half cycle. Each half cycle is full of additional modulations as predicted by theory. The similarity may be brought to closer by increasing the amplitudes of the resonance components $n \sim m\omega_1$.

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A NOTE ON THE PRINCIPAL RADII OF CURVATURE

By

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1. Sbrana, F., e Rollero, A* has called attention to an elementary property of the principal radii of normal curvature R_1, R_2 of a surface S, with $R_2 > R_1 > 0$. Taking a point P on the normal to S at O, at a numerical distance of r from O, and a point A on S near O, at a numerical distance of d from P, he has proved that d has a minimum value if $r \leqslant R_1$, has a maximum value if $r \geqslant R_2$ and has neither a maximum nor a minimum if $R_1 < r < R_2$. The object of this paper is to give independent proofs of these results analytically.

2. Let O be the point (x,y,z), A the point $(x+\delta x, y+\delta y, z+\delta z)$, and P the point $(x+rX, y+rY, z+rZ)$ where X, Y, Z are the direction cosines of the normal to the surface at O. Then

$$\begin{aligned} d^2 &= (\delta x - rX)^2 + (\delta y - rY)^2 + (\delta z - rZ)^2 \\ &= (\delta x^2 + \delta y^2 + \delta z^2) - 2r(X\delta x + Y\delta y + Z\delta z) + r^2 (X^2 + Y^2 + Z^2) \\ &= (Edu^2 + 2F\, du\, dv + Gdv^2) - 2r\Sigma[X. (x_1\, du + x_2 dv + \tfrac{1}{2} x_{11}\, du^2 + \\ &\qquad x_{12}\, du\, dv + \tfrac{1}{2} x_{22} dv^2)] + r^2, \end{aligned}$$

neglecting small quantities of the third and higher orders,

$$\text{or} \quad d^2 = E\, du^2 + 2F du\, dv + G dv^2 - 2r[\tfrac{1}{2} E du^2 + M du dv + \tfrac{1}{2} N dv^2] + r^2,$$

$$\text{since } \Sigma X x_1 = 0, \text{ and } \Sigma X x_2 = 0,$$

$$\text{or} \quad d^2 = (E - Lr) du^2 + 2(F - Mr)\, du\, dv + (G - Nr)\, dv^2 + r^2$$

where E, F, G; L, M, N are fundamental magnitudes of the first and second order.

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* Sbrana, F. *e* Rollero, A. Su una proprieta elementare dei raggi di curvatura delle curve *e* delle superficie, Atti Accad. Ligure 4 (1947), 16—20 (1948), but the periodical was not acces ible.

If the lines of curvature be taken as parametric curves, then $F=0$, $M=0$, and consequently the expression for d^2 becomes

$$d^2=(E-Lr)\,du^2+(G-Nr)\,dv^2+r^2.$$

3. Now d will have a stationary value if the expression $(E-Lr)du^2+(G-Nr)dv^2$ has a stationary value and the condition for this is,

$$(E-Lr).(G-Nr)\geqslant 0$$

i.e.

$$LNr^2-(EN+GL)r+EG\geqslant 0$$

The expression on the left equated to zero, gives the principal radii of curvature, which are real and positive by hypothesis. Also $LN-M^2>0$ since $R_1, R_2>0$. Therefore, the above inequality will hold good and d will have a stationary value if r lies outside the range (R_1, R_2), i.e., if $r\leqslant R_1$, or $r\geqslant R_2$, and will have no stationary value if $R_1<r<R_2$. Further the stationary value will be maximum, if $(E-Lr)$, $(G-Nr)$ not >0 and minimum if $(E-Lr)$, $(G-Nr)$ not <0.

4. If $r\leqslant R_1$, then $(E-Lr)\geqslant(E-LR_1)$, and $(G-Nr)\geqslant(G-NR_1)$. If $r\geqslant R_2$ then $(E-Lr)\leqslant(E-LR_2)$ and $(G-Nr)\leqslant(G-NR_2)$. Therefore for values of $r\leqslant R_1$, d will have a minimum value if $(E-LR_1)$, $(G-NR_1)\geqslant 0$. We shall see that both $(E-LR_1)$ and $(G-NR_1)$ cannot be simultaneously zero. Also for values of $r\geqslant R_2$, d will have a maximum value if $(E-LR_2)$, $(G-NR_2)\leqslant 0$. We shall see that both $(E-LR_2)$ and $(G-NR_2)$ cannot be simultaneously zero.

Now the equation giving the principal radii of curvature is

$$(LN-M^2)r^2-(EN-2FM+GL)r+(EG-F^2)=0$$

If the lines of curvature be taken as the parametric curves, then $M=0$, $F=0$, and also since both the radii of curvature are positive, $LN>0$.

The equation giving the principal radii of curvature becomes

$$LNr^2-(EN+GL)r+EG=0$$

Solving this equation for r, we have

$$R_1=\{(EN+GL)-\sqrt{(EN-GL)^2}\}/(2LN)$$
$$R_2=\{(EN+GL)+\sqrt{(EN-GL)^2}\}/(2LN).$$

Suppose $(EN-GL)>0$, then

$$R_1=\{(EN+GL)-(EN-GL)\}/(2LN)=G/N$$
$$R_2=\{(EN+GL)+(EN-GL)\}/(2LN)=E/L.$$

Therefore $L, N>0$ since $R_1, R_2>0$,
and $E-LR_1=E-LG/N=(EN-GL)/N>0$; $G-NR_1=G-NG/N=0$;
and $E-LR_2=E-LE/L=0$; $G-NR_2=G-NE/L=(LG-EN)/L<0$.

Suppose next $(EN-GL)<0$,
then $R_1=\{(EN+GL)-(GL-EN)\}/(2LN)=E/L$.
and $R_2=\{(EN+GL)+(GL-EN)\}/(2LN)=G/N$.
Therefore $(E-LR_1)=E-LE/L=0$; $(G-NR_1)=G-NE/L>0$
and $(E-LR_2)=E-LG/N<0$; $(G-NR_2)=G-NG/N=0$.
Also $(EN-GL)\neq 0$, since $NL>0$
Thus if $r\leqslant R_1$, the conditions for the existence of a mimimum value for d are satisfied, and if $r\geqslant R_2$, the conditions for the existence of a maximum value for d are satisfied.

Hence d has a minimum value, if $r\leqslant R_1$, a maximum value if $r\geqslant R_2$, neither a maximum nor a minimum value if $R_1<r<R_2$.

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